All Common Bipedal Gaits Emerge from a Single Passive Model

Zhenyu Gan, Yevgeniy Yesilevskiy, Petr Zaytsev, C. David Remy *

Abstract

In this paper, we systematically investigate passive gaits that emerge from the natural mechanical dynamics of a bipedal system. We use an energetically conservative model of a simple spring-leg biped that exhibits well-defined swing leg dynamics. Through a targeted continuation of periodic motions of this model, we systematically identify different gaits that emerge from simple bouncing in place. We show that these gaits arise along one-dimensional manifolds that bifurcate into different branches with distinctly different motions. The branching is associated with repeated breaks in symmetry of the motion. Among others, the resulting passive dynamic gaits include walking, running, hopping, skipping, and galloping. Our work establishes that the most common bipedal gaits can be obtained as different oscillatory motions (or nonlinear modes) of a single mechanical system with a single set of parameter values. For each of these gaits, the timing of swing leg motion and vertical motion is matched. This work thus supports the notion that different gaits are primarily a manifestation of the underlying natural mechanical dynamics of a legged system. Our results might explain the prevalence of certain gaits in nature, and may provide a blueprint for the design and control of energetically economical legged robots.

1. Introduction

In nature, legged animals use different gaits to locomote at different speeds, to prey, to escape danger, and to avoid obstacles (Alexander, 2003). Each individual gait follows a distinctly different motion pattern. It can be characterized by a specific footfall sequence (Alexander, 1984; Hildebrand, 1989), the typical ground reaction force profile (Alexander, 1980), or by how gravitational, potential, and kinetic energy are exchanged over the course of a stride (Cavagna et al., 1976). Despite vast differences in morphology, the gaits of many animals are strikingly similar (Alexander, 2013). Biped, such as humans and birds, prefer to walk at low speeds and to run at higher speeds. Quadrupedal mammals, across many species, walk, trot, and gallop. Some other gaits are less common: red kangaroos, for example, are known to hop (Dawson and Taylor, 1973), and some primates such as lemurs tend to leap at a large range of speeds (Demes et al., 1999; Franz et al., 2005). These less common gaits are also observed in humans in special situations such as fast turning, stairs descending, or in low gravity environments (Fiers et al., 2013; Minetti, 1998b; Pavei et al., 2015).

Experiments with humans and animals have shown that switching between different gaits as locomotion speed changes, enables more energetically economical locomotion. It has also been hypothesized that such energetic economy is the primary

* The authors are with the Robotics and Motion Laboratory (RAMlab), Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI (ganzheny@umich.edu, yevyes@umich.edu, pzaytsev@umich.edu, cdremy@umich.edu). This material is based upon work supported by the National Science Foundation under Grant No. 1453346. Additional funding for this project was provided by NIH (GRANT: 1-R01-EB019834-2014).
driving factor behind the choice of gait for many animals (Hoyt and Taylor, 1981; Minetti et al., 1999). In the world of robotics, simulation studies have corroborated this hypothesis by using numerical optimization to auto-generate energy optimal motions for dynamic models of legged systems (Mordatch et al., 2010; Srinivasan and Ruina, 2006). In our own work, we have performed such analysis for conceptual models of bipeds (Xi and Remy, 2014) and quadrupeds (Xi et al., 2016), as well as for a detailed model of the bipedal robot RAMone (Smit-Anseeuw et al., 2017). The energy optimal motions found in all these studies closely resemble the different gaits found in nature. For example, in our bipedal models, we found walking and running to be the most energetically economical. In our quadrupedal model, walking, trotting, and galloping were the most economical. Since these models only reflect the mechanical dynamics of the legged system, these studies point to a mechanistic explanation for why certain gaits are more energetically favorable than others.

However, being largely based on computer generated motions, the above studies cannot explain the appearance of distinct gaits on a more fundamental level: why are the same motion patterns found to be the most energetically favorable across very different animals, conceptual models, and legged robots? Why do the same motions emerge for all of these systems, and, why do we not see other varieties of economical locomotion? Being able to answer these fundamental questions paves a road towards a more systematic design of energetically economical legged robots.

In this paper, we seek to study these questions by investigating the idea that all gaits are a manifestation of the underlying natural passive dynamics of a legged system. Our work establishes that all common bipedal gaits can be obtained as different oscillatory motions, or modes, of a single mechanical system. To this end, we reduce the model of a bipedal system to a point where it becomes completely lossless, while carefully maintaining its dynamical properties. Such reduced model has neither actuation nor control, and can only move according to its inherent mechanical dynamics.

Using numerical continuation techniques (Allgower and Georg, 2003), we systematically identify all possible period-one periodic motions of this model that emerge from simple bouncing in place with certain footfall patterns. Periodic motions lie, as a function of energy, along one dimensional manifolds that bifurcate into different branches with distinctly different motions. We find that the model is able to reproduce almost all of the bipedal gaits found in nature and robotics by gradually breaking symmetries in periodic solutions. In addition to walking and running, these include several gaits not present in prior models, such as hopping, skipping, or bipedal galloping. All these gaits originate through bifurcations and continuations from in-place bouncing of a single model with a single set of parameter values.

Our work builds on a range of models of legged systems that have been developed to understand the underlying dynamics in legged systems. These models abstract complex systems with many degrees of freedom, substantial amounts of soft tissue deformation, and intricate neural control. A notable subset of these models look to explain the motions utilizing passive dynamics; that is, the natural motions arising from the interplay of gravity, inertia, and elasticity. Mochon and McMahon (1980) demonstrated that the passive dynamics of the Inverted Pendulum (IP) model, which does not use controllers and actuators, closely resemble bipedal walking. Similarly, the Spring Loaded Inverted Pendulum (SLIP) model (Blickhan, 1989; Full and Koditschek, 1999) has been used to explain running gaits (Farley et al., 1993). These simple models have been widely studied and used to predict the relationships among speed, stride length, and stride frequency in legged systems (Blickhan, 1989; Kuo, 2001). They also have been used as templates for the design (Collins et al., 2005; Rezazadeh et al., 2015), motion generation (Mordatch et al., 2010), and control (Hereid et al., 2014) of legged robots.

All these passive models are designed to explain only one specific gait. However, animals in nature are able, with a single structure, to locomote with different gaits across a range of velocities. In order to analyze these locomotion patterns and design similar motions for legged robots, it is necessary to find a single unified model to explain the underlying mechanics of all common gaits. An important step in this direction was taken in 2006: Geyer et al. (2006) extended the SLIP model by adding a second compliant leg. They were able to show that this model can explain the dynamics of both bipedal walking and running. By using this model, they could reproduce realistic Ground Reaction Forces (GRFs) as well as Center of Mass (COM) trajectories for both gaits. Their work suggests that these two gaits are different dynamical modes of the same system, oscillating at different energy levels.
Figure 1. In this paper, we extend the well-known SLIP model to include two legs with passive swing leg dynamics. By taking the limit of \( m \to 0 \), we eliminate collision losses at touch-down and create an energetically conservative model. We use continuation methods to identify all possible passive periodic motions of this model, as they emerge from bouncing in place. Solutions lie, as a function of energy, along one dimensional manifolds that bifurcate into different branches with distinctly different motions. In doing so, we can reproduce almost all of the bipedal gaits found in nature and robotics, showing that they could be a manifestation of the passive natural dynamics of a legged system.

While the Geyer SLIP model has been instrumental in understanding gait dynamics, it has a key simplifying assumption inherited from the simple SLIP: after a leg lifts off the ground, it instantaneously goes to a predefined angle of attack and remains there until the foot hits the ground again. This modeling choice is problematic for our purposes, as it represents a controller of infinite fidelity rather than a passive dynamic motion. Most importantly, the angle of attack must be actively adapted to enable different gaits. At the same time, one can find infinitely many periodic motions by changing the angle of attack, including motions with infinitely short, nonphysical swing times (Rummel et al., 2009b). Furthermore, since in the SLIP model leg swing happens instantaneously, the footfall timing of two or more independent legs requires additional considerations, such as the introduction of a phase timer for each leg (Gan and Remy, 2014). Similar to the angle of attack, the choices of timing parameters are arbitrary and represent control rather than dynamics. Finally, as there is a single angle of attack for both legs, the model can only produce walking and running footfall patterns. It is unable to replicate other bipedal gaits, such as bipedal skipping or galloping (leaping), in which the two legs require different angles of attack. In order to understand the passive dynamic motion underlying the different gaits, it is thus necessary that leg swing is also a function of the natural mechanical dynamics. The way this is done in this work is very similar to the method proposed by O’Connor (2009) in which a torsional spring is added at the hip joint. The legs are therefore subjected to passive oscillations during swing.

In the following sections, we introduce a fully passive model that fulfills the requirements stated above. We then present the results of a continuation study that allows us to discover all different gaits for this model that emerge from bouncing in place. We finish with a discussion detailing the significance of our results.

2. Methods

2.1. Model Description

The bipedal model used in this paper (Fig. 1), is an extension of the work by Geyer et al. (2006), and is similar to the model proposed by O’Connor (2009). It consists of a main body with mass \( M \) and inertia \( J \). The vertical and horizontal positions of the center of mass (COM) of the main body, and the main body’s pitch angle are given by the variables \( x, y, \) and \( \varphi, \)
respectively. Each of the two legs (with index \( i \in \{ l, r \} \)) is modeled as a massless linear spring with length \( l_i \), rest leg length \( l_o \), spring stiffness \( k_{\text{leg}} \), no damping, and a point mass \( m \) at the foot. Each leg is connected, through a frictionless hip joint (with joint angle \( \alpha_i \)), to the COM of the main body. Torsional springs with stiffness \( k_{\text{swing}} \) and no damping are added to these joints. The springs are uncompressed when the respective leg is pointing straight down. Rather than defining a value for \( k_{\text{swing}} \) directly, we prescribe a leg swing frequency \( \omega_{\text{swing}} \), and derive:

\[
k_{\text{swing}} = \omega_{\text{swing}}^2 m l_o^2.
\]

While this model is energetically conservative during flight, it would lose energy when a foot’s velocity is brought to zero in a touch-down collision. With such losses, it would be impossible to identify passive periodic gaits. To resolve this issue, we take the limit of \( m \) going towards zero (\( m \to 0 \)), similar to the method used in Garcia et al. (1998) and O’Connor (2009). This implies that \( k_{\text{swing}} \to 0 \) as the swing frequency \( \omega_{\text{swing}} \) remains unchanged. Furthermore, the inertia of the main body becomes very large compared to the inertia of the feet (\( J \gg m l_o^2 \)), such that we can neglect any main body pitch and assume \( \varphi \equiv 0 \). Finally, the legs exert forces onto the main body only when they are compressed in stance; during swing, the main body motion is independent of the leg motion (but not the other way around). Since the leg stiffness \( k_{\text{leg}} \) is finite, infinitely fast oscillations of the leg length (around \( l_o \)) would occur while the leg is in the air. For viability, we ignore these oscillations and assume that the leg length is constant during swing (\( l \equiv l_o \)).

2.2. Equations of Motion

Based on these assumptions, we develop the equations of motion (EOM) for our model. For the main body, they are:

\[
\ddot{x} = F_x / M, \tag{2a} \\
\ddot{y} = F_y / M - g, \tag{2b}
\]

where \( F_x \) and \( F_y \) are the components of the combined leg force \( \mathbf{F} \).

During the swing phase of each leg, the leg angle acceleration is given by

\[
\ddot{\alpha}_i = \frac{1}{l_o^2 m} \left( \dot{x} l_o^2 \cos (\alpha_i) - (g + \ddot{y}) m l_o \sin (\alpha_i) - k_{\text{swing}} \alpha_i \right), \tag{3}
\]

which, in the limit \( m \to 0 \), and by substitution of (1) and (2), becomes:

\[
\ddot{\alpha}_i = -\cos (\alpha_i) \frac{F_x l_o}{M l_o^2} - \sin (\alpha_i) \frac{F_y l_o}{M l_o^2} - \alpha_i \omega_{\text{swing}}^2. \tag{4}
\]

During stance of each leg, a kinematic constraint is introduced on the leg length, leg angle, and the associated velocities. We assume that there is no sliding of the foot during the stance phase and use the variable \( s_i \) to record the horizontal foot positions on the ground. With this, the leg angle and leg length during stance can be computed as:

\[
\alpha_i = \arctan \left( \frac{s_i - x}{y} \right), \tag{5} \\
l_i = \sqrt{(x - s_i)^2 + y^2}. \tag{6}
\]
The forces exerted by the stance legs are:

\[ F_i = k_{\text{leg}} \cdot (l_o - l_i), \]
\[ F_x = \sum_i -F_i \sin (\alpha_i), \]
\[ F_y = \sum_i F_i \cos (\alpha_i). \]  (7a, 7b, 7c)

The transition between stance and swing of the individual legs is monitored via zero crossings of two event functions, \( e_{i,1} \) and \( e_{i,2} \). A touch-down event happens when the height of the COM is equal to the vertical projection of the leg length. A foot leaves the ground (the lift-off event) when the leg reaches its rest length.

\[ \text{touch-down event: } e_{i,1} = l_o \cos(\varphi + \alpha_i) - y, \]
\[ \text{lift-off event: } e_{i,2} = \sqrt{(x - s_i)^2 - y^2} - l_o. \]  (8a, 8b)

At touch-down, we record the new foot position on the ground, which is given by

\[ s_i = x + l_o \sin (\alpha_i). \]  (9)

Also note that during touch-down, the leg velocities \( \dot{l}_i \) and \( \dot{\alpha}_i \) are discontinuous. During lift-off, \( \dot{\alpha}_i \) remains unchanged, while \( \dot{l}_i \) is set to 0.

It is important to note, that our model has no way to change its leg length during swing (as humans and animals do by bending their knees). Instead, the leg length is fixed to \( l_o \) during the swing phase. To simulate swing when the COM is so low that the foot has to go through the ground, we simply ignore some of the touch-down events. The choice of whether to enforce a touch-down or allow the leg to swing through, is somewhat arbitrary in that it does not follow from the model’s dynamics. As we demonstrate later in this paper, this choice plays an important role in the evolution of, and distinction between different gaits. The contact events we consider are either specified explicitly in Section 2.3, or follow uniquely from the continuation of existing solutions.

In order to keep the solutions general, we normalize all state and parameter values with respect to \( M, g, \) and \( l_o \) (Hof, 1996). The only parameters that we had to select for the proposed model were the leg stiffness \( k_{\text{leg}} = 20 Mg/l_o \) and the swing frequency \( \omega_{\text{swing}} = \sqrt{5/9/l_o} \). These dimensionless values are comparable to our existing hardware (Smit et al., 2017).

2.3. Gait Creation

We define a gait as a periodic motion in which all states except for the horizontal position \( x \) return to their original values after one full stride. We further assume that the stride begins at the instance of a right leg touch-down \( e_{r,1} \). This event provides the Poincaré section for a limit cycle analysis. The state at the beginning of the stride (at the Poincaré section) is \( \mathbf{X}^o = [\dot{x}_o, y_o, \dot{y}_o, \alpha_{1,o}, \dot{\alpha}_{1,o}, \alpha_{r,o}, \dot{\alpha}_{r,o}]^T \). Starting at this initial state (and setting the body horizontal position \( x_o = 0 \)), we obtain a trajectory over a full stride through a numerical integration that is terminated when the subsequent touch-down event of the right leg is registered. This process yields the Poincaré map \( P \) that maps the initial states \( \mathbf{X}^o \) to final states \( \mathbf{X}' \) at the end of the stride. A zero function \( \Phi : \mathbb{R}^7 \rightarrow \mathbb{R}^7 \) enforces periodicity:

\[ \Phi := P (\mathbf{X}^*) - \mathbf{X}^* = 0. \]  (10)
Gaits are defined by states $X^*$ at the Poincaré section that are solutions to this implicit equation. Since the system is energetically conservative, only six of the seven periodicity constraints are independent. This means that solutions to (10) evolve along one-dimensional manifolds on the Poincaré section.

Some bipedal gaits in nature, such as walking, running, and skipping, are classified as symmetrical gaits. That is, the legs on each side perform exactly the same motion, just 180° out of phase (Alexander, 1984). To simplify computation of such symmetrical gaits, we cut the simulation of these gaits in half and terminate the integration at left leg touch-down $e_{l,1}$. To obtain a full stride, we then repeat the first half of the motion but with the left and right legs switched. For the asymmetrical gaits (hopping and bipedal galloping) integration was continued to the next right leg touch-down event, simulating a full stride.

Beyond the definition of the continuous states, we also have to define which leg is initially in stance and which is in swing. Since the Poincaré section is defined as right leg touch-down, the right leg is always initially in stance. This leads to two possible cases: single stance, in which the left leg is initially in the air, and double stance in which it is on the ground. Each case leads to a distinct dynamic behavior of the system and also imposes a different number of constraints on the continuous states.

2.4. Continuation and Bifurcations

We find branches of periodic solutions by solving a 1D continuation problem (Dankowicz and Schilder, 2013). That is, given a known solution $X^*_n$, we numerically search for an adjacent solution $X^*_{n+1}$ on the same branch, and, by iteratively repeating this process, discover the entire branch. At each iteration, we constrain the next solution $X^*_{n+1}$ to be a fixed distance $d$ away from $X^*_n$ in the direction that is similar to the previous step. That is, we find $X^*_{n+1}$ by solving:

$$
\Phi (X^*_{n+1}) = 0,
\|X^*_{n+1} - X^*_n\| = d,
(X^*_{n+1} - X^*_n)^T (X^*_n - X^*_{n-1}) > 0.
$$

Solving (11) numerically is sensitive to having a good initial guess of $X^*_{n+1}$. We generate such a guess based on the Floquet analysis of the Poincaré map. Suppose that for a given periodic trajectory $X^*_n$, there exists another trajectory $X^*_{n+1}$ nearby. Then a small disturbance in the direction of $X^*_{n+1}$ remains unchanged after one stride (after such disturbance, the passive system stays on the new periodic trajectory). Mathematically, this means that one of the Floquet multipliers$^1$ $\lambda_{n,i}$ of the system at $X^*_n$ is equal to $+1$, and the corresponding eigenvector $v_{n,i}$ is approximately directed towards $X^*_{n+1}$. Therefore, we use $X^*_{n+1} = X^*_n \pm d \cdot v_{n,i}$ as the initial guess for numerically solving the problem (11). The step size $d$ was carefully tuned during the search in order to identify periodic solutions, especially at turning points.

The continuation algorithm searches for adjacent periodic solutions associated with a Floquet multiplier equal to $+1$ on the same branch. For every solution in this branch, this is a stationary eigenvalue and the corresponding eigenvector is tangential to the branch. However, at some points along the branch, there may be more than one unit-multiplier. This happens when at least one of the other Floquet multipliers crosses the unit circle with a real value of $+1$ (Dankowicz and Schilder, 2013). The additional unit multipliers at such a Saddle-Node (SN) bifurcation usually have eigenvectors that are not tangential to the original branch of solutions. Another distinct branch of solutions is thus connected to the original

$^1$ Floquet multipliers are eigenvalues of the Monodromy matrix $J$, which is the partial derivative of the Poincaré map $P$ with respect to the system states, evaluated at the periodic solution. $J$ provides linear approximations to the changes in small disturbances to the system state from stride to stride:

$$
J_n = \frac{\partial P}{\partial X} |_{X^*_n}.
$$
This study considers three fundamentally different footfall patterns (shown here together with the corresponding vertical ground reaction forces) that originate from different contact considerations at the moment of touch-down (a). By choosing different combinations of supporting legs, the following three footfall sequences emerge: (b) single stance (running), (c) double stance (hopping), and (d) the combination of the previous two cases which includes both double stance and single stance in alternation.

New branches of solutions usually signify qualitative changes in the periodic trajectories, such as different footfall sequences or other brakes in symmetry. As the present study focuses on most common bipedal gaits and their relationships to each other, other types of bifurcations, such as the Period-doubling (PD) or Hopf bifurcations, were not considered.

2.5. Start of the Search

The search for gaits was initiated with the most trivial motion that fulfills our requirements for a gait: bouncing in place with the lowest energy possible. The initial state for this motion is \( X_0 = [0, l_0, 0, 0, 0, 0, 0] \), there is no forward speed, both legs are pointing down and are motionless, and the vertical height is equal to \( l_0 \) with no vertical velocity. When started in this state, the model undergoes a purely vertical cycle of leg compression and decompression, followed by an infinitesimally short flight phase before reaching the initial state again. This oscillation can happen either on one leg or on both legs, depending on whether we assume the initial state \( X_0 \) to be in (right leg) single stance or in double stance. Based on this distinction, we define four different footfall patterns which lead to different types of gaits:

- **Symmetrical single stance.** The right leg enters stance while the left leg does not. In our simplified model this means that the left leg is allowed to go through the ground at the initial instant. After one vertical oscillation and an infinitesimally short flight phase, the roles of the left and right leg are switched (Fig. 2b).
- **Asymmetrical double stance.** Both legs are fully synchronized: they start in stance phase and lift off the ground at the same time. This is shown as the first half of the footfall pattern in Fig. 2c.
- **Symmetrical double stance.** Both legs are fully synchronized as in the previous case, but after one oscillation (at the next touch-down) the legs are switched. This gait is shown as the whole gait cycle in Fig. 2c. The switching of the legs
3. Results

In the following, we present the periodic motions identified by our continuation analysis for this model. Solutions are discussed according to the footfall sequences from which they originate and are reported by stating the initial state \( \mathbf{X}_0 \) from which the periodic motion evolves. These initial states are plotted in a three-dimensional projection of the state space (showing the vertical velocity \( \dot{y} \), horizontal velocity \( \dot{x} \), and the angle of the right leg \( \alpha_r \) at the Poincaré section) to visualize the evolution and connection of individual solution branches and to highlight bifurcation points. For some selected periodic motions, we also present a visualization of the motion\(^2\) and show the vertical ground reaction forces (vGRFs).

We numerically report the most important SN bifurcation points and explicitly state their eigenvectors. Due to the contact constraints, not all of the seven initial states can be chosen freely. Since the right leg is uncompressed at touch-down (which defines the Poincaré section), the initial vertical position is fully determined by the right leg angle, \( y_o = l_o \cos(\alpha_{r,o}) \), and the right leg angle rate \( \dot{\alpha}_{r,o} \) is fully determined by the COM position (i.e. the angle \( \alpha_{r,o} \)) and velocity according to eq. (5). In addition, in initial double stance, the left leg velocity \( \dot{\alpha}_{l,o} \) is also no longer a free variable and, similar to \( \dot{\alpha}_{r,o} \), is determined by the COM position and rate and the left leg angle. The reported states and eigenvectors thus have five dimensions for initial single stance and only four dimensions for initial double stance. At bifurcation points, we describe the multidimensional eigenspace to the eigenvalue of +1 by reporting a set of independent eigenvectors that are tangential to the resulting solution branches.

3.1. Symmetrical Single Stance

Periodic motions emerging from the symmetrical single stance footfall pattern are shown in Fig. 3. They originate from bouncing-in-place (point \( S \), red circle) and can be generally classified as running-in-place (branch \( \text{RP} \)) and different types of running forward/backward (branches \( \text{R1-R6} \)).

Running-in-Place (Branch \( \text{RP} \)): At the starting point \( S \) –the initial bouncing-in-place motion– the eigenvector \( \mathbf{v}_k \) corresponding to the eigenvalue of 1 is pointing in the \( \dot{y} \) direction (see Table 1), corresponding to adjacent solutions with different vertical velocities \( \dot{y} \) at touch-down, but otherwise unchanged behavior. Positive initial velocities \( \dot{y}_o > 0 \) are impossible, since they correspond to lift-off, violating the touch-down condition at the Poincaré section. The \( \text{RP} \) branch thus extends from \( \dot{y}_o = 0 \sqrt{g/l_o} \) towards negative vertical velocities, corresponding to running-in-place with increasingly longer flight phases with a higher apex height and stance phases with a greater spring compression (grey line in Fig. 3). The branch is limited to \( \dot{y}_o > -\sqrt{2gl_o/M - 2gl_o} = -4.24 \sqrt{g/l_o} \). At this point, the leg spring fully compresses during stance \( (l_i = 0) \), leading to a singularity in eq. (5). Note that Fig. 3 only shows part of this branch for \( \dot{y}_o > -2 \sqrt{g/l_o} \). Along the \( \text{RP} \) branch, we find 13 SN bifurcations. The first five (points \( A-E \) in Fig. 3) are reported in Table 1.

Running Forward and Backward (Branches \( \text{R1-R6} \)): The additional eigenvectors with an eigenvalue of +1 at bifurcations \( A-E \) all have components in the \( \dot{x}, \dot{\alpha}_r, \) and \( \dot{\alpha}_l \) directions. The new solutions that stem from them thus incorporate horizontal velocity and leg swing, and correspond to running forward/backward gaits. The resulting branches (labeled \( \text{R2-R6} \) extend

\(^2\) Please see the multimedia attachment for a video showing all these motions.
Figure 3. Visualization of periodic motions that emerge from bouncing-in-place (red circle, \( S \)) with a symmetrical single stance (as shown in Fig. 2b). Solutions can be classified into running-in-place gaits with different vertical heights (grey line, \( RP \)), and running forward/backward gaits (\( R1 - R6 \)) which differ primarily by the number of swing leg oscillations in the associated solutions. Black circles (\( A - G \)) denote SN bifurcations that connect the solution branches. The numbers on the grey isolines denote the largest leg compression in a stride as a percentage of leg length. The inserts show the ground reaction forces of a sample running motion along \( R2 \) (c), as well as the transition to grounded running (when the flight phase disappears) along \( R1 \) (a) & (b). Several successive frames of solution (c) are presented in Fig. 4.

symmetrically for both negative and positive velocities \( \dot{x} \) (only the positive velocity range is shown in Fig. 3). Moving along these branches, maximal leg compression during stance (shown as isolines in Fig. 3) increases until the branches terminate in the singularity of a full leg compression.

Motions along the branch \( R2 \), which extends from bifurcation point \( A \), are most reminiscent of bipedal running gaits in nature. A representative motion (solution labeled (c)) on this branch is shown in Fig. 4. Trajectories of the leg angles \( \alpha \) essentially follow a quasi-sinusoidal profile (shown for a forward speed of \( 0.5 \sqrt{gL} \) in Fig. 5, red lines). For motions along \( R2 \), after lift-off, the swing leg first moves backward, then swings forward to be in front of the main body, and retracts backward before hitting the ground. Such ‘swing leg retraction’ is also observed in human running (Seyfarth et al., 2003), and, has the beneficial property that the foot closely matches the ground-speed at touchdown.

The periodic motions on the other running branches differ primarily in when the swing leg hits the ground. Motions along the branches \( R3 \) and \( R4 \) represent running gaits in which the swing leg performs two oscillations during swing (blue and green lines in Fig. 5). The solutions differ in when the touch-down event is triggered: on \( R3 \), the swing leg touches down while moving forwards (hence the large jump in velocity in Fig. 5b), while on \( R4 \) it hits the ground while moving backwards (as for motions on \( R2 \)). The solutions on the next branches (\( R5 \) and \( R6 \)) are analogous to those on \( R3 \) and
Table 1. Initial states and eigenvectors associated with a Floquet multiplier of $+1$ at selected bifurcation points for motions emerging from Symmetrical Single Stance.

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<th>R5</th>
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</tr>
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</table>

Switch legs to finish the stride cycle

Figure 4. Key frames from an exemplary solution on the running branch $R2$ (solution (c) in Fig. 3). Solutions on the other running branches have the same footfall pattern, but differ in their number of swing leg oscillations.

$R4$, but with the swing leg doing three oscillations during the flight phase. Overall, bifurcations along the $RP$ branch arise whenever the hopping period is matched by the oscillation time of the swing leg or a multiple thereof. Many combinations are possible as flight time scales linearly with $\dot{y}_{0}$, while the swing leg has a constant frequency. Further solutions follow this pattern with even higher numbers of swing leg oscillations (up to 7 oscillations in our model). These solutions are not explicitly shown and discussed here.

In Fig. 3, one can see that the branches $R5$ and $R6$ join at another SN bifurcation (point $G$) at high leg compression values. At this point, the swing leg velocity $\dot{\alpha}$ becomes zero at the moment of touchdown, thus essentially removing the difference between the motions in $R5$ and $R6$. Such a junction is not present for $R3$ and $R4$, as these two branches reach full leg compression before joining together. However, we can find an SN bifurcation (point $F$) along $R2$ that corresponds to such a junction. It connects to another branch of running gaits, $R1$, in which the swing leg performs a single oscillation and the foot strikes the ground in a forward motion (purple lines in Fig. 5). Solutions on the branch $R1$ have the same footfall pattern as running in $R2$, but shorter flight phases and lower COM profiles. At lower speeds on the $R1$ branch, the aerial phase becomes shorter (see solution labeled (b) in Fig. 3) until it fully vanishes at a velocity of $0.25 \sqrt{gl_{0}}$. As speed further decreases, the touch-down of the leading leg happens earlier than the lift-off of the trailing leg and the flight phase is replaced by a double stance phase (see solution labeled (a) in Fig. 3). These double-stance gaits are not running (there is no flight phase); but they retain a single-hump vGRF profile, similar to that of running. Therefore, motions like this have been referred to as grounded running gaits ($GR$) (Martinez and Carbajal, 2011; Rummel et al., 2009b). This type of solution only exists when the forward speed is larger than $0.24 \sqrt{gl_{0}}$. For lower speeds, the $GR$ branch would require
Figure 5. In this figure, we compare the right leg angle trajectory $\alpha_r(t)$ and right leg angle velocity $\dot{\alpha}_r(t)$ of solutions on the running branches ($R1 - R4$). For each solution, we show a full stride with an average forward velocity of $0.5 \sqrt{\frac{g}{l}}$. Stance phases are shown as dotted lines, while solid lines indicate flight phases. The solutions start and end at right leg touch-down. One can clearly see how the solutions differ in their number of oscillations and whether the swing leg hits the ground in a forward or backward motion. These differences are enabled by different durations of the flight phase.

negative vGRFs during the double stance phase which is not physically realistic. This is the reason why the $R1/GR$ branch does not connect to the $RP$ branch.

3.2. Asymmetrical Double Stance

The periodic motions emerging from the asymmetrical double stance footfall pattern are shown in Fig. 6. Again, they originate from bouncing-in-place (point $S$, red circle) and can be generally classified as hopping-in-place (branch $HP$), hopping forward/backward (branches $H1$ and $H2$), galloping-in-place (branches $GP1$ and $GP2$), and galloping (branches $G1$ and $G2$).

Hopping-in-place (Branch $HP$): Starting from bouncing-in-place on two legs, the eigenvector $v_S$ corresponding to the eigenvalue of $+1$ is pointing in the $\dot{y}$ direction (see Table 2). By varying the vertical speed $\dot{y}$ at touch-down, we find the hopping-in-place branch $HP$ (grey line in Fig. 6). In this gait, the two legs are fully synchronized and always vertical. The $HP$ branch extends to $\dot{y}_o = \sqrt{\frac{2k_{ext}l_o}{M} - 2g l_o} = -6.16 \sqrt{\frac{g}{l}}$, where the legs fully compress during stance. This value is larger than for the symmetrical single stance, as now both legs provide forces in the stance phase. (Note again that Fig. 6 only shows part of this branch for $\dot{y}_o > -2\sqrt{\frac{g}{l}}$). We find 16 SN bifurcations along the $HP$ branch. The first two SN bifurcations (points $I$ and $J$) identified along the $HP$ branch are pointing to the forward/backward hopping branches. They are at the vertical speeds $-0.70 \sqrt{\frac{g}{l}}$ and $-1.11 \sqrt{\frac{g}{l}}$, respectively. The corresponding eigenvectors are provided in Table 2. As with the running gaits, 6 additional hopping branches with multiple periods of swing leg oscillations appear in pairs along $HP$ (not shown in the figure). The remaining SN bifurcations, including the next two bifurcations that are found at the vertical speeds $-1.19 \sqrt{\frac{g}{l}}$ and $-1.41 \sqrt{\frac{g}{l}}$ (points $P$ and $Q$), are the starting points of galloping-in-place gaits.
For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Galloping (Branches $G_1$ and $G_2$): As provided in Table 2, at bifurcation $L$, in addition to moving along the hopping branch $H_2$ (direction $v_{J_2}$) where the legs are fully synchronized, there exist additional solutions in direction $v_{I_2}$, away from $H_2$. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. For these new solutions, the leg motions become desynchronized: one leg slightly moves backward and becomes the trailing leg. Table 2. Initial states and eigenvectors associated with a Floquet multiplier of +1 at selected bifurcation points for motions emerging from Asymmetrical Double Stance.

<table>
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<tr>
<th>States</th>
<th>S</th>
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<th>P</th>
<th>Q</th>
<th>L</th>
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Figure 6. Visualization of periodic motions that emerge from bouncing-in-place (red circle, $S$) with an *asymmetrical double stance* (as shown in Fig. 2c). Solutions can be classified into hopping-in-place gaits with different vertical heights (grey line, $HP$), hopping forward/backward gaits ($H1$ & $H2$), galloping-in-place gaits ($GP1$ & $GP2$), and galloping gaits ($G1$ & $G2$). The black circles ($I$, $J$, $L$, $N$, $P$ & $Q$) denote SN bifurcations that connect the solution branches. The inserts show the ground reaction forces of a typical hopping gait along the $H2$ branch (d), as well as the transition from galloping (e) to an asymmetrical running gait (f) as an additional flight phase appears along $G1$. Key motion frames from the solutions (d) and (e) are shown in Fig. 7 and Fig. 8.

Leg, while the other leg moves forward acting as the leading leg. Either leg can be the leading leg which corresponds to moving in either the positive or negative direction of $v_{L2}$ (right leg in front: branch $G1$; left leg in front: branch $G2$). The gaits along $G1$ and $G2$ are referred to as *bipedal galloping*. An exemplary solution labeled (e) of this gait is shown in Fig. 8. Compared to the running and hopping gaits, the two legs have distinct angles at touch-down and follow different trajectories. Since each leg swings forward to its original angle after each stride, the leading and the trailing legs remain the same along the galloping branches. At high speeds, both $G1$ and $G2$ revert back to the hopping branch $H2$ and merge with it at another SN bifurcation (point $N$), where the legs become synchronized again.

Close to the point $L$, galloping is similar to hopping, with only a slight desynchronization between the motion of the two legs. Moving away from $H2$, the legs become less synchronized, with an increasing difference in their angles of attack. The double stance phase becomes shorter and eventually disappears. It is replaced by a brief flight phase after the speed reaches $3.31 \sqrt{gl}$ (see solution (f) in Fig. 6). Such galloping with two distinct flight phases can be thought of as *asymmetrical running*: it has the same footfall pattern as a running gait, but different touch-down angles of the two legs and different step durations. This gait is shown as the dashed line portion in the middle of $G1$ in Fig. 6. Closer to the point $N$, the additional flight phase gets shorter again and is eventually replaced again by a double stance phase.
Finished one stride cycle, switching legs to finish the second stride

**Figure 7.** Key frames from an exemplary solution on the hopping branch $H2$ (solution (d) in Fig. 6). In this hopping forward/backward gait, the two legs are always fully synchronized and act essentially as a single (yet stiffer) leg.

Next stride

**Figure 8.** Key frames from an exemplary solution on the hopping branch $G1$ (solution (e) in Fig. 6). In such a galloping gait, the trailing leg always touches down on the ground first. The ground contact of the leading leg follows shortly after. In contrast to the skipping gait, the leading leg and trailing leg never switch roles. Galloping is thus an asymmetrical gait. However, both the left leg and right leg can serve as the leading leg, corresponding to the left galloping $G1$ and right galloping $G2$ branches.

### 3.3. Symmetrical Double Stance

The periodic motions emerging from symmetrical double stance are shown in Fig. 9. The solution branches include hopping-in-place $HP$ and hopping forward/backward $H1$ and $H2$, identical to those presented in Section 3.2. In addition, we find skipping gaits along the new branches $SP1$, $SP2$, $S1$, and $S2$.

**Hopping (Branches $HP$, $H1$, and $H2$):** As long as the motion of the two legs is synchronized, gaits emerging from symmetrical double stance remain the same after leg switching and are thus identical to those emerging from asymmetrical double stance, as discussed in Section 3.2. Therefore, starting from symmetrical bouncing at $S$, the hopping branches $HP$, $H1$, and $H2$ are found again at the same locations. However, we find a new bifurcation point $H$ on the $HP$ branch, when the vertical speed $\dot{y}_o$ equals $-0.58 \sqrt{g_h}$. Also, at the point $I$ ($\dot{y}_o = -0.70 \sqrt{g_h}$) where the $H1$ branch connects to $HP$, two additional Floquet multipliers cross the unit circle at $+1$ with the corresponding eigenvectors $v_{I3}$ and $v_{I4}$ reported in Table 3. That is, two additional branches connect to $HP$ at point $I$.

**Skipping-in-place (Branches $SP1$ and $SP2$):** At points $H$ and $I$ periodic solutions $SP1$ and $SP2$ bifurcate from the hopping-in-place branch $HP$. Their corresponding eigenvectors $v_{H2}$ and $v_{I3}$ (see Table 3) indicate that these two branches extend in the same direction: both leg angles increase equally in magnitude but in opposite directions, while the vertical speed $\dot{y}$
and the horizontal speed $\dot{x}$ remain the same. Solutions on these branches are bouncing-in-place motions in which the two legs move symmetrically with opposite angles and velocities. During swing, the front leg moves backward and hits the ground in the back, and vice versa. If the roles of the left and right leg are switched (right leg is initially behind and left leg in front), the branches extend in the negative $v_{H2}$ and $v_{I3}$ directions (not fully shown in the figure). Several frames of a typical motion on $SP1$ are illustrated in Fig. 10. As the leg angle $\alpha_I$ at touch-down increases, the duration of the flight phase also increases. Both $SP1$ and $SP2$ terminate when they reach the unstable equilibrium point in the middle of the double-support phase, when the sum of the leg forces is equal to the weight of the main body, $Mg = 2k_{leg} \cos(\alpha_I)(l_o - l_i)$.

Similar to the running gaits $R1$ and $R2$, the branches $SP1$ and $SP2$ differ in how the swing leg hits the ground: in $SP1$ the leg moves forward just before touch-down and in $SP2$ backward. Also, just like for the running gaits, additional skipping branches with multiple periods of swing leg oscillations appear in pairs along $HP$ which are neither shown nor discussed here.

Along the two branches, solutions have roughly the same vertical speeds and the COM only moves in the vertical direction with $\dot{x} = 0$ during the whole stride cycle. We find 5 SN bifurcation points that emerge along the $SP1$ and $SP2$ branches. Most of them lead to skipping gaits with unequal vGRFs and these branches eventually become single-leg hopping gaits. They are neither shown nor discussed here. Only one of them leads to a gait with the same maximal vGRFs in both legs and has a continuation to a walking gait. This is shown in Fig. 9 at point $K$ (Table 3). We describe this gait $S2$ below.

**Table 3.** Initial states and eigenvectors associated with a Floquet multiplier of $+1$ at selected bifurcation points for motions emerging from Symmetrical Double Stance.

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</table>

| Branch | $HP$ | $HP$ | $SP1$ | $HP$ | $HI$ | $SP2$ | $S1$ | $HP$ | $H2$ | $SP1$ | $S2$ | $HI$ | $SI$ | $4H$ | $4W$ |

**Forward/Backward Skipping (Branch S1):** At point $I$, there is another eigenvalue of $+1$, associated with the eigenvector $v_{I4}$. Hence, another set of solutions $SI$ emerges from the hopping-in-place branch $HP$. Along $v_{I4}$ the leg angles change in opposite directions but, in contrast to $v_{I3}$, by different amounts. This difference in the initial angle magnitudes results in the COM moving forward/backward. The footfall pattern of this gait is demonstrated in Fig. 11 and it includes all possible combinations of leg phases (double stance, right leg single stance, left leg single stance, and flight). This gait is known as skipping (Minetti, 1998a). In contrast to the hopping gaits where the legs move simultaneously, in skipping one leg moves faster than the other, causing the legs to hit the ground with different angles of attack. The two legs alternately act as the leading and trailing leg. As a result, skipping gaits are represented by two lines in Fig. 9, the solid $SI$ line (right leg in front) and the dashed $SI$ line (left leg in front), and each particular motion ‘jumps’ between them. At low speeds, skipping gaits from $SI$ are similar to the skipping-in-place gait $SP2$. As the speed increases, however, touch-down angles of both legs increase to positive values, but the trailing leg angle is always smaller than that of the leading leg (see Fig. 9, solution labeled (h)). When the forward speed reaches $3.74 \sqrt{\frac{g}{l_o}}$, the double stance phase is replaced by a short flight phase as shown in Fig. 9, solution (i). As we further increase the forward speed, this brief flight phase disappears and phases of the legs tend to synchronize again. Eventually, $SI$ connects back to the hopping branch $HI$ at point $M$, as shown in Fig. 9. It
Figure 9. Visualization of periodic motions that emerge from bouncing-in-place (red circle, $S$) with a symmetrical double stance (as shown in Fig. 2c). Solutions include the same hopping gaits observed in Fig. 6, as well as a number of skipping gaits. The branches $SP1$ and $SP2$ correspond to skipping-in-place with split legs. Branch $S1$ corresponds to a forward skipping gait. The solid line represents this gait as started from the touch-down event of the leading leg whereas the dashed line represents the same gait as started from the touch-down of the trailing leg. Branch $S2$ is one of many examples of a forward skipping gait that emerges from in-place skipping. It eventually connects to solutions that can be classified as walking. The black circles ($H$ - $K$ & $M$) denote bifurcations that connect the solution branches. The inserts show the ground reaction forces of a typical hopping gait along the $H1$ branch (g), as well as the transition from skipping (h) to an asymmetrical running gait (i) as an additional flight phase appears along $S1$. Several key frames from the solution (h) are shown in Fig. 11.

should be noted that as the forward speed becomes negative, this branch extends symmetrically to the opposite side on the Poincaré section. In these solutions, the roles of left and right leg are simply switched and the model skips backwards.

Transition to Walking (Branch $S2$): A similar skipping gait $S2$ bifurcates from the branch $SP1$ at point $K$, as shown in Fig. 9. For solutions along $S2$, both leg angles increase, but at different rates, and both the horizontal and vertical speed magnitudes increase (see eigenvector $v_{K2}$ in Table 3). In $S2$ gaits, the footfall sequence is exactly the same as in $S1$. Swing legs rotate away from the COM and retract back to the center before hitting the ground. At low speeds, the legs are fully synchronized. As the forward speed grows, the duration of the flight phase decreases and the phase delay between the legs gets bigger. Shortly after reaching the fastest speed $0.462 \sqrt{gl_0}$ along $S2$, the flight phase disappears. The branch thus vanishes at the solution labeled (o) (see Fig. 13) when the two adjacent single stance phases of the same leg merge together. A single stance phase with two maxima in the leg forces appears as a natural smooth continuation of $S2$ skipping. The resulting double-humped vGRF motion is the most common walking gait, which we discuss below along with other types of walking that we found.
Figure 10. Key frames from the first half of a typical solution (at bifurcation point $K'$ in Fig. 9) on the skipping-in-place $SP1$ branch. In this gait, the two legs always have the same angles and angular velocities but move in opposite directions. Both legs touch down and leave the ground at the same time. During the flight phase, the roles of trailing leg and leading leg are switched. This is a symmetrical gait and it thus takes another hop to finish a full stride cycle.

Figure 11. Key frames from the first half of a typical skipping gait on the $SJ$ branch (solution (h) in Fig. 9). This gait starts in the double stance phase, with the right leg acting as the leading leg. During the aerial phase, the right leg moves back and becomes the trailing leg. It takes another step for the right leg to move forward and finish the whole stride cycle. In order to find this gait, the two legs need to have different angles at their respective touch-down.
3.4. Symmetrical Single/Double Stance

4-Beat Hopping (Branch 4H): Following the footfall pattern shown in Fig. 2d at point S and varying the vertical speed $\dot{y}$, a hopping-in-place gait including both a single stance phase and a double stance phase is found. Fig. 13 shows that at point O, the same footfall pattern can be used to locomote the COM forward/backward at low speeds (branch 4H). An exemplary motion of the 4H gait (solution (j) in Fig. 13) is shown in Fig. 12. As the horizontal speed increases, the durations of the flight phases in this gait get shorter. Eventually, the three adjacent stance phases of the same leg are joined together as a prolonged stance phase, leading to the same footfall sequence as in walking. This happens when the horizontal velocity reaches $0.04 \sqrt{gl_o}$ (see solution (k) in Fig. 13).

Triple-humped Walking: This prolonged-stance walking gait is called triple-humped walking. As shown in an exemplary solution (l) in Fig. 13, each leg undergoes three compressions in the entire stance phase: one compression in each of the two double stance phases and a third compression during the single stance in between. At higher speeds, as the stance phase gets shorter, these oscillations become less pronounced and the peak forces approach the body weight $Mg$. Eventually all oscillations along the stance leg settle down to just one compression.

Level Walking: The single compression during stance of such walking gait results in a prolonged, single humped shape of the vGRFs (see solution (m) in Fig. 13). This gait is sometimes referred to as level walking (Rummel et al., 2009b). It has the smallest vertical excursion of the COM trajectory, $0.93 - 0.95 l_o$, compared to all other walking gaits.

Double-humped Walking: Starting from the level walking, as the forward speed increases, we found the most human-like walking gait (Pandy, 2003). This walking gait has two maxima in the leg forces during stance (Fig. 13, solution labeled (n)) and a local minimum in the leg forces appears at exactly the middle of the single stance phase. In contrast to all other gaits in which the COM moves to the lowest position (maximum leg compression) at mid-stance, in the double-humped walking gait, the COM vaults to the highest point when the leg is vertical (see Fig. 14). As we increase the forward speed, the leg force tends to decrease in single stance. When the speed gets to $0.52 \sqrt{gl_o}$, the minimal leg compression becomes zero with no forces in the stance leg (see solution (o) in Fig. 13). A flight phase appears as a natural continuation after passing this solution, and the gait smoothly transitions to the symmetrical skipping S2. Thus, the double-humped walking spans only a small range of speeds $0.27 - 0.45 \sqrt{gl_o}$ in our passive model.
Figure 13. Visualization of periodic motions that emerge from bouncing-in-place (red circle, S) with a symmetrical single/double stance (as shown in Fig. 2d). A forward/backward speed is introduced at bifurcation point O, and at solution (k) all air phases vanish, leading to walking with triple-hump ground reaction forces. This gait undergoes a substantial change in its ground reaction force profile and gradually transitions to level walking, and eventually walking with double-hump ground reaction forces. At solution labeled (o), an additional lift-off arises and the walking gait smoothly changes to the previously identified skipping gait on branch S2. The inserts show how the ground reaction forces profiles evolves along this branch. Several successive motion frames of 4-beat hopping (j) and double-humped walking (n) are shown in Fig. 12 and Fig. 14.

4. Conclusion and Discussion

In this paper, we systematically investigated passive dynamic gaits that emerge from the natural mechanical dynamics of a bipedal legged system. To this end, we developed an energetically conservative, yet dynamically complete model of a biped. We achieved this by extending the established SLIP model to include two legs and by adding a foot mass and a hip spring to enable passive swing leg dynamics. By letting the foot mass and hip stiffness go to zero while keeping their ratio (and thus the leg swing frequency $\omega_{\text{swing}}$) constant, we prevented energy losses at touchdown. Through a targeted continuation of periodic motions, we showed that a range of different bipedal gaits emerged in this model from a simple bouncing-in-place motion with different discrete footfall patterns. Among others, these passive dynamic gaits included walking, running, hopping, skipping, and galloping. The different gaits arose along one-dimensional manifolds of solutions. These manifolds bifurcated into different branches with distinctly different types of motions (as shown in Figs. 3, 6, 9, and 13). That is, the gaits were obtained as different oscillatory motions (or nonlinear modes) of a single mechanical system with a single set of parameters.

As our biped model has neither actuation nor control, our work supports our hypothesis that different gaits are primarily a manifestation of the underlying natural mechanical dynamics of a legged system. The occurrence and prevalence of certain gaits in nature are thus possibly the consequence of animals exploiting passivity based gaits in order to move in an energetically economical fashion. The same argument should hold for legged robots: the passive motions derived in this
paper establish a blueprint of how to move economically. In the absence of losses, the passive dynamic gaits constitute the only feasible way of locomoting without performing any actuator work. As losses are introduced, such as losses due to friction and collision impacts, the motions will have to change and will, of course, require some actuation. However, staying close to the original passive motions might reduce the need for motor torques and for negative actuator work, and might hence reduce the energetic cost of locomotion.

On some level, we can consider all gaits found in this paper to originate from a bipedal model that is completely at rest, standing either on one or both legs with the main body supported by a lightly compressed leg spring that balances gravity. One can think of this equilibrium as a state of maximal symmetry. It is invariant with regards to any time shift (temporal symmetry), to switching the forward and backward directions (symmetrical to the frontal plane), and (for the case of double support) to switching of the left and right legs (symmetrical to the sagittal plane). Different gaits arise from this equilibrium through a sequence of breaks in symmetry that occur at bifurcations (Stewart and Golubitsky, 2010). The first is the introduction of a vertical oscillation that breaks the temporal symmetry and introduces a periodic motion. This oscillation is linear for small amplitudes. However, beyond a certain point, the legs will fully extend, and the model will enter a flight phase which renders the dynamics nonlinear. This is the point where our study begins to track the motion as either running (RP) or hopping (HP, H4) in place. The gaits that branch out from these motions further reduce the symmetry. For example, hopping gaits (H1, H2) break the frontal plane symmetry, while the left and right legs still can be switched without changing the motion. This sagittal plane symmetry is then broken in another bifurcation as the galloping gaits (G1, G2) emerge. A similar process leads to all the other gaits. Skipping-in-place (SP1, SP2), for example, has a mixed symmetry: solutions remain invariant to switching both forward/backward and left/right at the same time. Again, this symmetry is broken as soon as forward motion is introduced. This is shown, for example, in the S2 branch. A similar consideration holds for the running gaits (R1-R6) which introduce a forward/backward motion into an already left/right asymmetric running-in-place RP. The breaks in symmetry happen whenever the vertical motion of the model is matched to the leg swing. Since the swing legs are undergoing quasi-sinusoidal motions dictated by the torsional spring at the hip, the same leg angles can be reached with a different number of oscillation periods. This creates several branches of the same gait, such as the running branches R2, R4, and R6 as illustrated in Fig. 5. Furthermore, contact can happen either in a forward or backward motion of the swing leg. All gaits, including all of the running gaits R1-R6, hopping forward/backward H1-H2, and skipping in place SP1-SP2, thus appear in pairs.

From the perspective of a roboticist, the most exciting aspect of this work lies in its future potential as a design tool. The proposed approach can be used directly to develop energetically economical motions that exploit the natural dynamics of the system. It might also explain, why our past optimization studies (Smit-Anseeuw et al., 2017; Xi and Remy, 2014; Xi et al., 2016) have converged to motions that closely resemble gaits found in nature. Furthermore, the approach could be extended to be used in the development of new hardware that incorporates suitable dynamic elements. These dynamics...
could be tuned on a simplified model to yield beneficial natural dynamics that enable economical locomotion on a full robot. Our results also suggest that swing leg motions in bipedal systems can be simplified to be a nearly passive spring oscillation which has the same natural frequency for all bipedal gaits at different speeds. The fact that the model relies on a single set of parameters for all motions, also provides some reasoning that fixed stiffness actuators might be a sufficient choice for legged robots and that variable stiffness actuation is not necessarily required to achieve energetic economy. From the perspective of biology, we have shown that this unified model can reproduce most of the common bipedal gaits that are observed in nature. The motions obtained from our model seem qualitatively similar to gaits of animals. This similarity includes the footfall sequences that are characteristic of each gait, the shape of the vertical ground contact forces, as well as details such as swing leg retraction. For example, our model can produce a double-humped ground reaction force profile along the W branch of solutions that is characteristic of walking in nature (Pandy, 2003). Yet, while our approach provides a reason why many bipeds in nature use similar gaits (and do not locomote in arbitrarily different ways), it cannot explain why a certain biped walks at slow speeds, runs at high speeds, and reverts to hopping, skipping, and galloping only in very special situations. In order to explain these choices among the passivity-based gaits, we would need to be able to explain which of the provided motions are more efficient at a given speed in systems that have losses. This is obviously not possible in a passive model, and would require further considerations about the energetic impact of each motion.

As a further limitation, the gaits found in this study do not necessarily constitute the complete set of all possible periodic motions of this model. While we can say definitively that the continuation/bifurcation approach finds all period-one gaits that originate from bouncing in place with the given footfall patterns, there might be other gaits that do not connect to these starting motions, or that connect to them under conditions that are not considered valid in our study. An example is the R1 running branch, that would connect to RP only if negative vGRFs were allowed and that was only detected through continuation, because it also connects to the R2 branch. It is possible that other branches exist, but remain undetected as they suffer from similar limitations. Other examples that are omitted from this study include motions with different footfall sequences, such as single leg hopping. Furthermore, in this study, we limited ourselves to trace only SN bifurcations. Other types of bifurcation exist, and could lead to other types of gaits. For example, we found that asymmetrical running gaits with different angles of attack at each step emerge as period-doubling (PD) bifurcations from the period-one running gaits R1-R6. On these branches of period-two solutions, further running gaits with periods of four steps are found. As the PD bifurcations keep emerging, they eventually lead to chaotic effects that are similar to those found in passive dynamic walkers (Garcia et al., 1998). Including all these solutions was simply out the scope of this work, yet could provide an interesting avenue for further explorations. For all results presented here, we used a fixed parameter set for our model. In particular, we had to make a somewhat arbitrary choice about how to select the leg stiffness and swing leg frequency. It remains an open question how our results may or may not change for different parameter values. Hence, another natural continuation of our work is a parameter study that systematically tracks how locations of the bifurcation points and branches on the Poincaré section change. This information would be particularly valuable from a design point of view. In a preliminary parameter study, we have already found that changes in parameter values can induce structural changes in our solution landscape. For example, when increasing \( \omega_{\text{swing}} \), the grounded running GR and walking W branches merge together at about \( \omega_{\text{swing}} = \sqrt{35 \sqrt{g/l_o}} \). For higher swing frequencies, grounded running and level walking become the same gait and a smooth transition from walking to running can be observed. This is similar to the result by (Rummel et al., 2009b), who reported that there was no speed gap between running and walking.

Our work consolidates a number of previous efforts to explain different gaits through mechanical dynamic models (Blickhan, 1989; Farley et al., 1993; Full and Koditschek, 1999; Geyer et al., 2006; Mochon and McMahon, 1980; Rummel et al., 2009b). To the best of our knowledge, this is the first time it has been shown that a single model with fixed parameters can generate all common bipedal gaits at once. Furthermore, our results provide a new point of view on the relationships among different gaits, as we show that they are all connected to bouncing in place through breaks in symmetry. Our model therefore has the potential to serve as a general template (Full and Koditschek, 1999) for the study of bipedal locomotion.
Previous models were either only able to produce a single gait (such as passive dynamic walking (McGeer, 1990)) or relied on a (control) parameter that had to be adjusted to obtain different motions (such as the angle of attack in the traditional SLIP model (Geyer et al., 2006; Rummel et al., 2009b)). In the case of the Geyer SLIP model, for example, a series of similar solutions can be found for the same angle of attack by varying the total energy in the model (Geyer et al., 2006; Rummel et al., 2009a). Yet, the value of the angle of attack has to change for the same gait at various speeds and it has to take on different values to enable the two gaits, walking and running, that the Geyer SLIP model can exhibit (Rummel et al., 2009b). Moreover, since these models are essentially controlled, infinitely many solutions can be produced by gradually changing the angle of attack, leading to a high-dimensional solution space with no clear boundaries on individual gaits. In contrast, in the present work, the different gaits are found along distinct one-dimensional manifolds that are only connected through a series of discrete bifurcations. That is, while our results extend the range of gaits that can be explained by the mechanical dynamics of a single model, they also define these gaits in a more precise manner. For the individual running branches, for example, the solutions found within our study form a well-defined subset of the solution range of the SLIP model. In particular, they contain only solutions that can be achieved with passive swing dynamics and thus exclude, for example, motions that require unreasonably fast leg swing or are otherwise not physically achievable with an actual legged system. Due to the distinct nature of individual solution branches, our work thus has the potential to provide a new definition of a gait: a distinct nonlinear dynamic mode of the underlying passive mechanical system.

References


